A Database of Elliptic Curves with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ torsion

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Motivation and goals of the project

Main goals:

- Calculating the average rank of a family of elliptic curves with torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.
- Finding elliptic curves with torsion subgroup $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ and rank 4.
- Comparing the behaviour of the convergence of the average rank of this family to other families.

Motivation

We consider elliptic curves E over $\mathbb{Q}$ with defining equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

with $a_i \in \mathbb{Q}$ and $\Delta(E) \neq 0$.

What is the probability that an elliptic curve has infinitely many rational points?

This can be stated as the following limit:

$$P = \lim_{X\to\infty} \frac{\#\{\text{elliptic curves with cond.} \leq X \text{ and infinite rational points}\}}{\#\{\text{elliptic curves with cond.} \leq X\}}.$$

Minimalist Conjecture

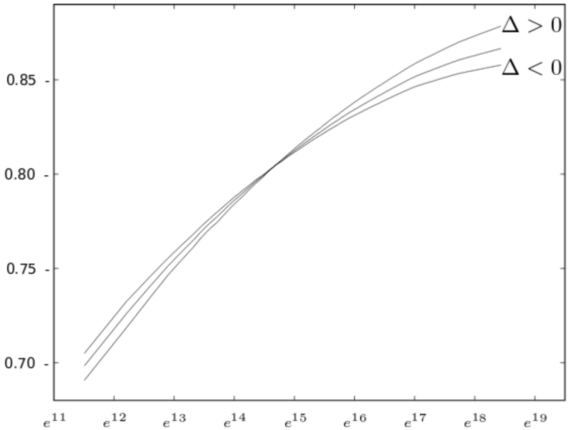
$$P = 1/2.$$

Bhargava, Shankar When all elliptic curves over $\mathbb{Q}$ are ordered by height, their average rank is at most 0.885.

Tables of elliptic curves

- Cremona table
- Stein-Watkins table

In 2006, B. Bektemirov, B. Mazur, W. Stein and M. Watkins did some calculations for the average rank of elliptic curves for conductor $\leq 10^8$.



A turnaround

Instead of ordering elliptic curves by conductor, what happens if we order them by height? Given an elliptic curve E over $\mathbb{Q}$ with short Weierstrass equation

$$y^2 = x^3 + a_4 x + a_6,$$

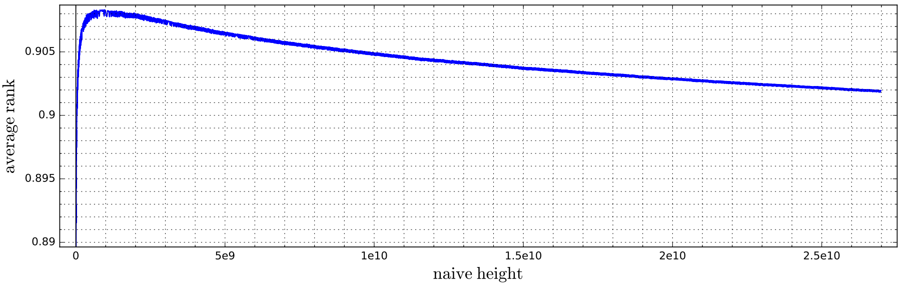
we define the (naive) height of E as

$$H(E) := \max\left\{4|a_4|^3, 27a_6^2\right\}.$$

Wht would the average rank look like under this ordering of elliptic curves? First step: make a database.

Average rank of elliptic curves up to a given height

In 2016, J. Balakrishnan, W. Ho, N. Kaplan, S. Spicer, W. Stein and J. Weigandt created an exhaustive database of isomorphism classes of elliptic curves with naive height up to $2.7 \cdot 10^{10}$, for a total of $238,764,310$ curves.



Families of curves

What happens to average rank for families of elliptic curves with extra structure?

In our case, we considered the family of elliptic curves

$$y^2 = x(x+1)(x+u^4),$$

with $u = 2t/(t^2-1)$, $t = a/b$, $a, b \in \mathbb{Z}$. This family generically has torsion $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

The height of an elliptic curve in this family is given by $\max\{a, b\}$.

Rank-computing functions

Sage functions

- John Cremona's `mwrank`.
- The function `analytic_rank_upper_bound`.

Magma functions

- `MordellWeilShaInformation`
- Tom Fisher's `TwoPowerIsogenyDescentRankBound`

Creating the database

- Step 1: Enumerate all curves up to a given height.
- Step 2: Divide these curves into chunks and run parallel computations on them.
- Step 3: Combine all of this information under the assumption that GRH, BSD and parity are true.

Difficulties

The computation often involves curves with large conductors (up to height 150 we already have conductors $> 10^{21}$).

- Computing the analytic rank with higher precision is slow and takes up a lot of memory. Increasing the precision further eventually runs into a PARI error.
- Running `MordellWeilShaInformation` with higher effort runs into errors.
- `TwoPowerIsogenyDescentRankBound` is able to reduce the upper bound for the rank fairly quickly. (Average of about 3 curve/ min.)
- Eliminates more than half of the previously undetermined curves.
- More practical for large curves in improving upper bounds compared to running `analytic_rank_upper_bound`.

Higher Descents on an Elliptic Curve With a Rational 2-Torsion Point (Tom Fisher, 2017)

Magma function: `TwoPowerIsogenyDescentRankBound`

e.g. Compute rank of an elliptic curve with minimal model $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ where $[a_1, \cdots, a_6] =$ $[1, 0, 0, -135768822388741791302652934970985712783554707\,90,$ 608903107189124544552375122006463375051783003664916991371908093947492]

Sage function `mwrank` gives a lower bound of 0 and an upper bound of 2. Fisher's function determined that the actual rank of this curve is 0.

The ideas behind Tom Fisher's function

$$
\begin{tikzcd}
 & D_2 \arrow[r, "\pi_4"] \arrow[d] & D_1 \arrow[r, "\pi_3"] \arrow[d] & D \arrow[r, "\pi_2"] \arrow[d] & C \arrow[d] \arrow[dr, "\pi_1"] & \\
\cdots \arrow[r] & E' \arrow[r, "\hat{\phi}"] \arrow[rr, bend right, "{[2]}"'] & E \arrow[r, "\phi"] & E' \arrow[r, "\hat{\phi}"] \arrow[rr, bend right, "{[2]}"'] & E \arrow[r, "\phi"] & E'
\end{tikzcd}
$$

$$rank E(\mathbb{Q}) = \dim \frac{E'(\mathbb{Q})}{\phi E(\mathbb{Q})} + \dim \frac{E(\mathbb{Q})}{\hat{\phi} E'(\mathbb{Q})} - \dim E(\mathbb{Q})[\phi] - \dim E'([\hat{\phi}])$$

$$\frac{E(\mathbb{Q})}{\hat{\phi} E'(\mathbb{Q})} \subset \cdots \subset S_2 \subset S_1$$

$$\frac{E'(\mathbb{Q})}{\phi E(\mathbb{Q})} \subset \cdots \subset S'_2 \subset S'_1$$

Rank computed

Before this week:

height	# curves	# determined	% determined
$0 \leq H < 100$	2000	1985	91.15
$100 \leq H < 200$	66617	4248	69.91
$200 \leq H < 300$	10125	5939	58.66
$300 \leq H < 400$	14134	8066	57.07
$400 \leq H < 500$	18230	9858	54.08
$500 \leq H < 600$	22306	12021	53.89
$600 \leq H < 700$	26296	13859	52.70

Rank computed

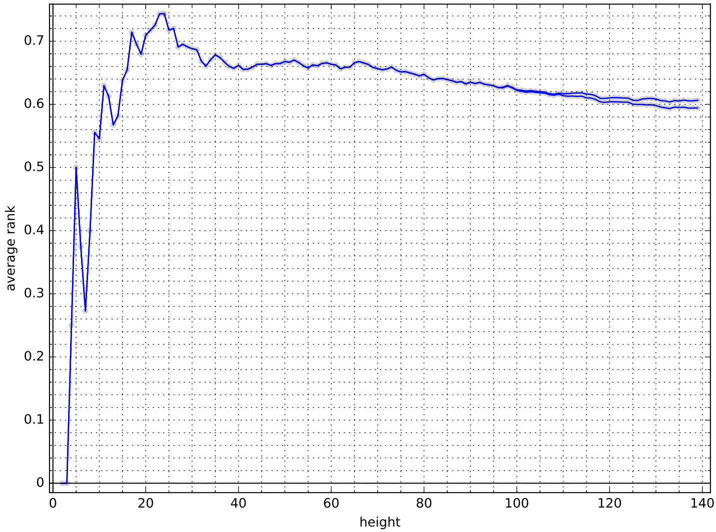
After this week:

height	# curves	# determined	% determined
$0 \leq H < 100$	2000	1999	99.95
$100 \leq H < 200$	6076	5822	95.82
$200 \leq H < 300$	10125	9278	91.63
$300 \leq H < 400$	14134	12961	91.70
$400 \leq H < 500$	18230	16617	91.15
$500 \leq H < 600$	22306	20375	91.34
$600 \leq H < 700$	26296	23852	90.71

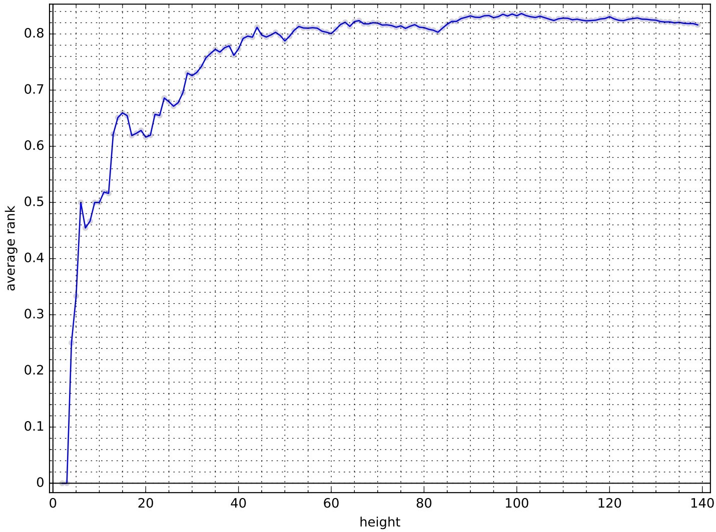
Results

- No curves with rational torsion subgroup $\mathbb{Z}/2 \times \mathbb{Z}/8$ and rank 4 exist with height < 1000.
- The average rank of the curves with rational torsion subgroup $\mathbb{Z}/2 \times \mathbb{Z}/8$ and height < 140 lies between 0.594 and 0.606.
- The average rank of the curves with rational torsion subgroup $\mathbb{Z}/2 \times \mathbb{Z}/8$ and height < 700 lies between 0.508 and 0.679.

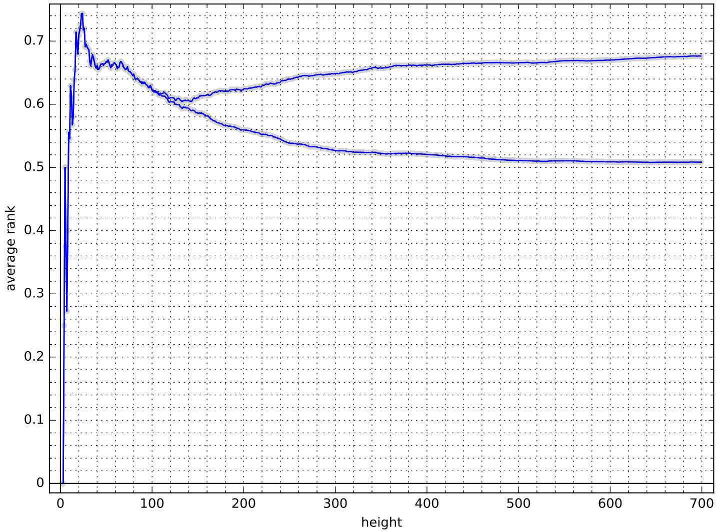
The average rank of curves with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$-torsion



The average rank of curves with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$-torsion



The average rank of curves with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$-torsion



Open questions/future work

- Do we see similar behaviour in other families? (8-torsion, 12-torsion)
- Is this behaviour caused by the torsion subgroup of order 16? Or is it true for any curves with similar height?
- Use the databases to compute Szpiro's constant (cf. S. Anni's group); current record we have is 6.94 (world record is 9.02)

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